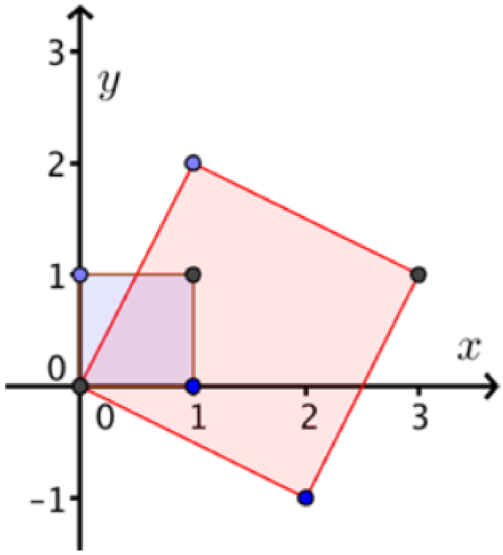
Solutions to short-answer questions

1 a We let $x = 2$ and $y = 3$ so that
$(2,3) \to (2 \times 2 + 3, -2 + 2 \times 3) = (7,4)$.

b The matrix of the transformation is given by the coefficients in the rule, that is,
$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

c The fastest way to find the image of the unit square is to evaluate
$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$$
The columns then give the required points:
$$(0,0), (2,-1), (1,2), (3,1)$$
The square is shown in blue, and its image in red.



Since the original area is 1, the area of the image will be $\textbf{Area} = |ad - bc| = |2 \times 2 - 1 \times (-1)| = 5$

d Since the matrix of this linear transformation is
$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$
the inverse transformation will have matrix
$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{2 \times 2 - 1 \times (-1)} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \end{aligned}$$
Therefore, the rule of the inverse transformation is $(x,y) \to \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

c $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

e

$$\begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

f $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3 a Since

$$\tan\theta = 3 = \frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 5 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{3^2+1^2} = \sqrt{10}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = \frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1 \\ &= \frac{2}{10} - 1 \\ &= -\frac{4}{5}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\frac{1}{\sqrt{10}}\frac{3}{\sqrt{10}} \\ &= \frac{3}{5}. \end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

b The image of the point $(2, 4)$ can be found by evaluating,

$$\frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 22 \end{bmatrix}.$$

Therefore, $(2, 4) \to \left(\frac{4}{5}, \frac{22}{5}\right)$.

4 a The matrix that will reflect the plane in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The matrix that will reflect the plane in the line $y = -x$ is given by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

b The matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The dilation matrix by a factor of 2 from the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}.$$

c The matrix that will reflect the plane in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrix that will skew the result by a factor of 2 from the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

5 a

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} x-3 \\ -y+4 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \to (x-3, -y+4)$.

b

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x-3 \\ y+4 \end{bmatrix} \\ &= \begin{bmatrix} x-3 \\ -y-4 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \to (x-3, -y-4)$.

6 a The required matrix is

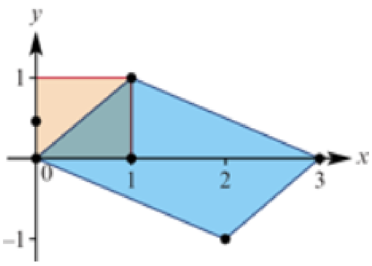
$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{1 \times 1 - 0 \times k}\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}. \end{aligned}$$

This matrix will shear each point in the y-direction by a factor of $-k$.

7 a The unit square is shown in red, and its image in blue.



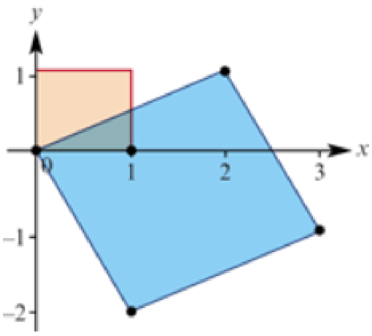
The determinant of this linear transformation is

$\det B = 2 \times 1 - 1 \times (-1) = 2 + 1 = 3.$

The unit square has area 1 square unit, so to find the area of its image we evaluate:

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= 3 \times 1 \\ &= 3 \text{ square units.} \end{aligned}$$

b The unit square is shown in red, and its image in blue.



The determinant of this linear transformation is

$\det B = 2 \times (-2) - 1 \times 1 = -4 - 1 = -5.$

The unit square has area 1 square unit, so to find the area of its image we evaluate:

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= |-5| \times 1 \\ &= 5 \text{ square units.} \end{aligned}$$

8 a We do this as a sequence of three steps:

- translate the plane so that the origin is the centre of rotation.
- rotate the plane about the origin by 90° anticlockwise.
- translate the plane back to its original position.

Firstly the rotation matrix is

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the overall transformation of

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y-1 \\ x-1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y \\ x-2 \end{bmatrix} \end{aligned}$$

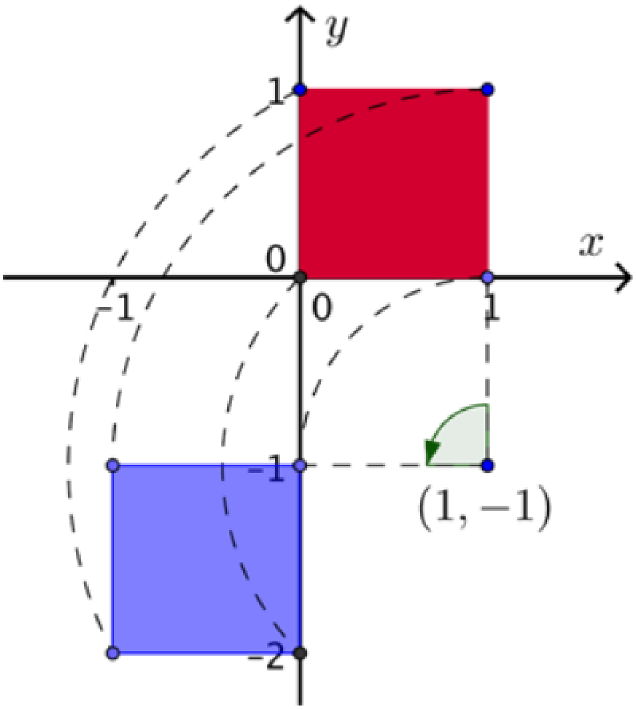
b To find the image of the point $(2, -1)$. Let $x = 2$ and $y = -1$ so that

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -y \\ x-2 \end{bmatrix} \\ &= \begin{bmatrix} -(-1) \\ 2-2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore $(2, -1) \to (1, 0)$.

c

The unit square is shown in red, and its image after the rotation is in blue.



Solutions to multiple-choice questions

1 **B** The point $(2, -1)$ maps to the point

$$(2 \times 2 - 3 \times (-1), -2 + 4 \times (-1)) = (7, -6).$$

2 **D** The required transformation is $(x, y) \to (-y, -x)$, which corresponds to matrix

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

3 **A** The matrix that will dilate the plane by a factor of 2 from the y-axis is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix that will reflect the result in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

4 **D** The location of the negative entry suggests that this should be a reflection matrix. Indeed, if $\theta = 30^\circ$ then,

$$\begin{aligned}&\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\\ =&\begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix}\\ =&\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.\end{aligned}$$

This corresponds to a reflection in the line $y = x\tan 30^\circ$.

5 **C** Firstly, matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the required transformation is given by

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)\\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x+2 \\ y-3 \end{bmatrix}\\ &= \begin{bmatrix} -y+3 \\ x+2 \end{bmatrix}\end{aligned}$$

Therefore, the transformation is $(x, y) \to (-y+3, x+2)$.

6 **A** Note that this matrix is equal to the product:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This corresponds to a rotation by 180° (or, equivalently, a reflection through the origin) followed by a dilation by a factor of 2 from the x-axis.

7 **D** Note that this matrix corresponds to a reflection in both the x and y axes. So we draw the graph of $y = (x-1)^2$, then reflect this in each axis. Alternatively, you can show that the transformed graph has equation $y = -(x+1)^2$.

8 **E** We simply need to find the matrix that has a determinant of 2. Only the last matrix has this property.

9 **D** Matrix R is a rotation matrix of 40°. Therefore, matrix R^n is a rotation matrix of $40m^\circ$. Since a rotation by any multiple of 360° corresponds to the identity matrix, we need to find the smallest value of m such that $40m$ is a multiple of 360°. Therefore, $m = 9$.

Solutions to extended-response questions

1 **a** The required rotation matrix is

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

b The required rotation matrix is

$$\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

c A 45° rotation followed by a 30° rotation will give a 75° rotation. Therefore, the required matrix is

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} \end{bmatrix}.$$

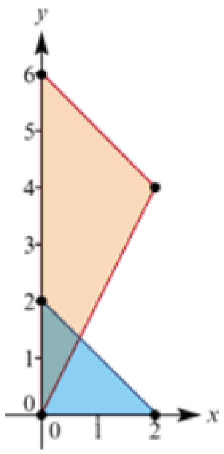
d The rotation matrix of 75° is also given by the expression

$$\begin{bmatrix} \cos 75^\circ & -\sin 75^\circ \\ \sin 75^\circ & \cos 75^\circ \end{bmatrix}.$$

Comparing the entries of these two matrices gives

$$\cos 75^\circ = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4},$$
$$\sin 75^\circ = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

2 **a**



The triangle is shown in blue and its image in red.

b The area of the original triangle is

$$\frac{bh}{2} = \frac{2 \times 2}{2} = 2.$$

Therefore the area of the image will be given by,

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= |1 \times 3 - 0 \times 2| \times 2 \\ &= 3 \times \frac{1}{2} \\ &= 6 \text{ square units.} \end{aligned}$$

c When the red figure is revolved around the y-axis, we obtain a figure that is the compound of two cones. The upper cone has base radius $r_1 = 2$ and height $h_1 = 2$. The lower cone has base radius $r = 2$ and height $h = 4$. Therefore, the total volume will be

$$\begin{aligned} V &= \frac{1}{3}\pi r_1^2 h_1 + \frac{1}{3}\pi r_2^2 h_2 \\ &= \frac{1}{3} \times \pi 2^2 \times 2 + \frac{1}{3} \times \pi 2^2 \times 4 \\ &= 8\pi \text{ cubic units.} \end{aligned}$$

3 a The matrix of the transformation is obtained by reading off the coefficients in the rule for the linear transformation. That is,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

b This transformation is a shear by a factor of 1 in the x direction.

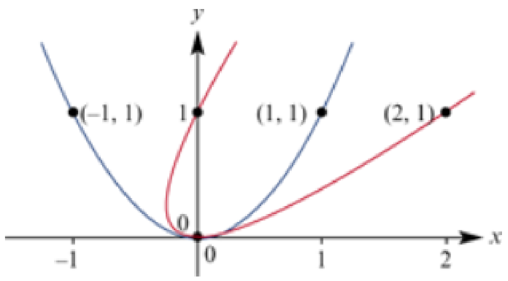
c The image of the points can be found in one step by evaluating,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The columns then give the required points:

$$(0,0), (2,1), (0,1).$$

d The image will be a sheared parabola, shown in red. The original parabola is shown in blue.



4 a The matrix of the transformation is

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

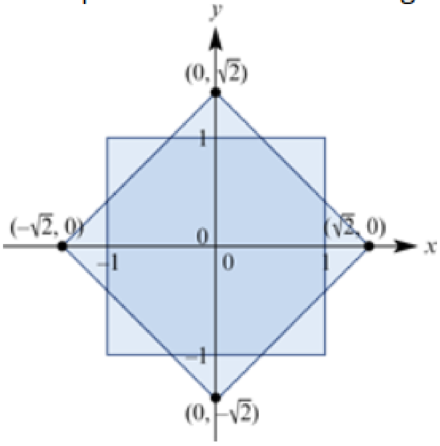
To find the image of the point $(1,1)$ we multiply,

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}.$$

Therefore $(1,1) \to (0,\sqrt{2})$. Since this matrix will rotate the square by 45° anticlockwise, the four points must be:

$$(0,\sqrt{2}), (\sqrt{2},0), (0,-\sqrt{2}), (-\sqrt{2},0).$$

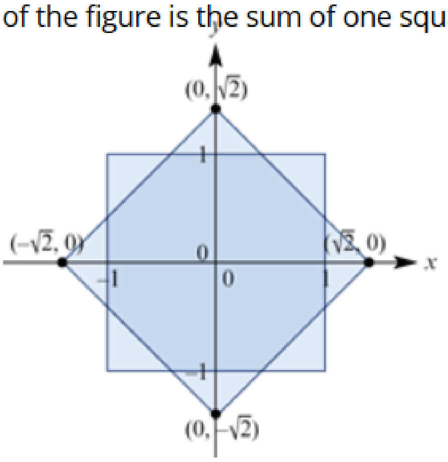
b The square and its rotated image are shown below.



c The area of the shape can be found in many ways. We will find the coordinates of point A shown in the above diagram. Point A is the intersection of the lines

$$y = 1 \text{ and } x + y = \sqrt{2}.$$

Solving this pair of equations gives $x = \sqrt{2} - 1$ and $y = 1$ so that the required point is $A(\sqrt{2} - 1, 1)$. The area of the figure is the sum of one square and four triangles, one of which is indicated in red below.



Since point A has coordinates $(\sqrt{2} - 1, 1)$, the area of each triangle is

$$\begin{aligned} A &= \frac{bh}{2} \\ &= \frac{(2\sqrt{2} - 2)(\sqrt{2} - 1)}{2} \\ &= 3 - 2\sqrt{2}. \end{aligned}$$

Therefore, the total area will be $A = 1 + 4 \times (3 - 2\sqrt{2})$

$$= 13 - 8\sqrt{2} \text{ square units.}$$

5 a i

$$\begin{aligned} &\mathrm{Rot}(\theta)\mathrm{Rot}(\phi) \\ =&\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \\ =&\begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\theta) \\ \sin\theta\cos\phi + \cos\theta\sin\theta & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} \\ =&\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \\ =&\mathrm{Rot}(\theta+\phi) \end{aligned}$$

ii $\text{Ref}(\theta)\text{Ref}(\phi)$

$$=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos 2\phi & \sin 2\phi\\ \sin 2\phi & -\cos 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta\cos 2\phi+\sin 2\theta\sin 2\phi & -(\sin 2\theta\cos 2\phi-\cos 2\theta\sin 2\theta)\\ \sin 2\theta\cos 2\phi-\cos 2\theta\sin 2\theta & \cos 2\theta\cos 2\phi+\sin 2\theta\sin 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta-2\phi) & -\sin(2\theta-2\phi)\\ \sin(2\theta-2\phi) & \cos(2\theta-2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\theta-\phi)) & -\sin(2(\theta-\phi))\\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi))\end{bmatrix}$$

$$=\text{Rot}(2(\theta-\phi))$$

iii $\text{Rot}(\theta)\text{Ref}(\phi)$

$$=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos 2\phi & \sin 2\phi\\ \sin 2\phi & -\cos 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta\cos 2\phi-\sin\theta\sin 2\phi & \sin\theta\cos 2\phi+\cos\theta\sin 2\theta\\ \sin\theta\cos 2\phi+\cos\theta\sin 2\theta & -(\cos\theta\cos 2\phi-\sin\theta\sin 2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\theta+2\phi) & \sin(\theta+2\phi)\\ \sin(\theta+2\phi) & -\cos(\theta+2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\phi+\theta/2)) & \sin(2(\phi+\theta/2))\\ \sin(2(\phi+\theta/2)) & -\cos(2(\phi+\theta/2))\end{bmatrix}$$

$$=\text{Ref}(\phi+\theta/2)$$

iv $\text{Ref}(\theta)\text{Rot}(\phi)$

$$=\begin{bmatrix}\cos 2\phi & \sin 2\phi\\ \sin 2\phi & -\cos 2\phi\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta\cos\phi+\sin 2\theta\sin\phi & \sin 2\theta\cos\phi-\cos 2\theta\sin\theta\\ \sin 2\theta\cos\phi-\cos 2\theta\sin\theta & -(\cos 2\theta\cos\phi+\sin 2\theta\sin\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta-\phi) & \sin(2\theta-\phi)\\ \sin(2\theta-\phi) & -\cos(2\theta-\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\theta-\phi/2)) & \sin(2(\theta-\phi/2))\\ \sin(2(\theta-\phi/2)) & -\cos(2(\theta-\phi/2))\end{bmatrix}$$

$$=\text{Ref}(\theta-\phi/2)$$

b **i** The composition of two rotations is a rotation.

ii The composition of two reflections is a rotation.

iii The composition of a reflection followed by a rotation is a reflection.

iv The composition of a rotation followed by a reflection is a reflection.

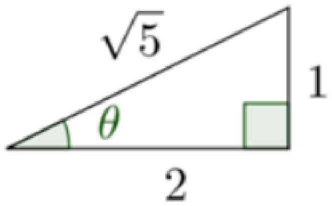
c Evaluating from left to right we have,

$$\begin{aligned}
&\mathrm{Rot}(60^\circ)\mathrm{Ref}(60^\circ)\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
=&(\mathrm{Rot}(60^\circ)\mathrm{Ref}(60^\circ))\,\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
=&\mathrm{Ref}(60^\circ+30^\circ)\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
=&\mathrm{Ref}(60^\circ)\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
=&(\mathrm{Ref}(60^\circ)\mathrm{Ref}(60^\circ))\,\mathrm{Rot}(60^\circ)\\
=&\mathrm{Rot}(2(90^\circ-60^\circ))\mathrm{Rot}(60^\circ)\\
=&\mathrm{Rot}(60^\circ)\mathrm{Rot}(60^\circ)\\
=&\mathrm{Rot}(120^\circ)\\
=&\begin{bmatrix}\cos 120^\circ & -\sin 120^\circ\\ \sin 120^\circ & \cos 120^\circ\end{bmatrix}\\
=&\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}
\end{aligned}$$

6 a Since

$$\tan\theta = \frac{1}{2},$$

we draw a right angled triangle with opposite and adjacent lengths 1 and 2 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{5}$.



Therefore,

$$\cos\theta = \frac{2}{\sqrt{5}} \text{ and } \sin\theta = \frac{1}{\sqrt{5}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1 = 2\left(\frac{2}{\sqrt{5}}\right)^2 - 1 = \frac{8}{5} - 1 = \frac{3}{5},\\
\sin 2\theta &= 2\sin\theta\cos\theta = 2\frac{1}{\sqrt{5}}\frac{2}{\sqrt{5}} = \frac{4}{5}.
\end{aligned}$$

Therefore the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}\end{bmatrix}.$$

b The image of the point $A(-3,1)$ is found by evaluating the matrix product,

$$\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}\end{bmatrix}\begin{bmatrix}-3\\ 1\end{bmatrix} = \begin{bmatrix}-1\\ -3\end{bmatrix}.$$

Therefore, the required point is $A'(-1,-3)$.

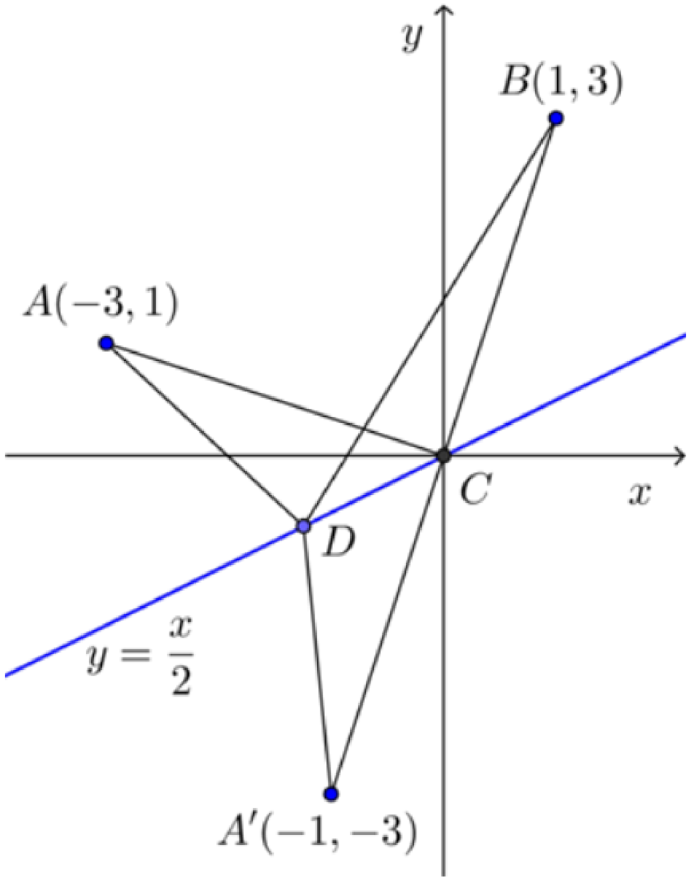
c Using the distance formula we find that

$$\begin{aligned} A'B &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(1-(-1))^2+(3-(-3))^2} \\ &= \sqrt{2^2+6^2} \\ &= \sqrt{40} \\ &= 2\sqrt{10}. \end{aligned}$$

d The line $y = \frac{x}{2}$ is the perpendicular bisects of line AA'. Therefore, $CA = CA'$, so that triangle ACA' is isosceles.

e Referring to the diagram below we have:

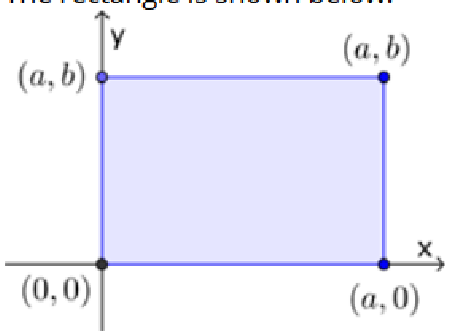
$$\begin{aligned} AD + DB &= A'D + DB \quad \text{(triangle } ADA' \text{ is isosceles)} \\ &> A'B \quad \text{(the side length of a triangle is always less than the sum of the other two)} \\ &= A'C + CB \\ &= AC + CB \quad \text{(triangle } ACA' \text{ is isosceles)} \end{aligned}$$



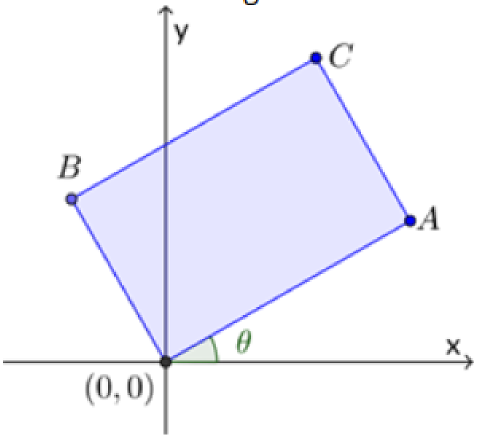
f The above calculation shows that $AC + CB$ is the shortest distance from A to B via the line. Therefore the shortest distance is

$$AC + CB = A'C + CB = A'B = 2\sqrt{10}.$$

7 a The rectangle is shown below.



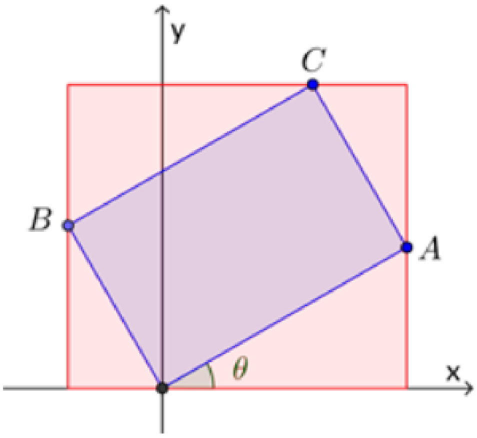
b The rotated rectangle is shown below.



We apply the rotation matrix to the coordinate of the original rectangle to find the following co-ordinates:

$$A(a\cos\theta, a\sin\theta),$$
$$B(-b\sin\theta, b\cos\theta),$$
$$C(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta).$$

c The rectangle described is shown in red in the diagram below.



Using coordinates A, B and C found in the previous question, we can find the area of the triangle. Its width is equal to

$$a\cos\theta + b\sin\theta,$$

and its height is equal to

$$a\sin\theta + b\cos\theta.$$

Therefore, its area is

$$\begin{aligned} A &= (a\cos\theta + b\sin\theta)(a\sin\theta + b\cos\theta) \\ &= a^2\cos\theta\sin\theta + ab\cos^2\theta + ab\sin^2\theta + b^2\cos\theta\sin\theta \\ &= (a^2+b^2)\cos\theta\sin\theta + ab(\cos^2\theta + \sin^2\theta) \\ &= (a^2+b^2)\cos\theta\sin\theta + ab(\cos^2\theta + \sin^2\theta) \\ &= (a^2+b^2)\cos\theta\sin\theta + ab \\ &= \frac{(a^2+b^2)}{2}\sin 2\theta + ab \end{aligned}$$

d For θ between 0 and $90°$, the maximum value of $\sin 2\theta$ occurs when $\theta = \frac{\pi}{4}$. Therefore, the maximum area will be

$$\begin{aligned} A &= \frac{(a^2+b^2)}{2} + ab \\ &= \frac{(a^2+2ab+b^2)}{2} \\ &= \frac{(a+b)^2}{2}, \end{aligned}$$

as required.

8 a Line L_1 is perpendicular to the line $y = mx$ and so has gradient $-\frac{1}{m}$. Moreover, it goes through the point $(1, 0)$. Therefore, its equation can be easily found:

$$\begin{aligned} y - 0 &= -\frac{1}{m}(x-1) \\ y &= -\frac{x}{m} + \frac{1}{m} \\ &= \frac{1}{m} - \frac{x}{m}. \end{aligned}$$

To find where the line intersects the unit circle, we substitute $y = \frac{1}{m} - \frac{x}{m}$ into the equation for the circle, $x^2 + y^2 = 1$ and solve. This gives,

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(\frac{1}{m} - \frac{x}{m}\right)^2 &= 1 \\ x^2 + \frac{1}{m^2} - \frac{2x}{m^2} + \frac{x^2}{m^2} &= 1 \\ m^2x^2 + 1 - 2x + x^2 &= m^2 \\ (m^2+1)x^2 - 2x + (1-m^2) &= 0. \end{aligned}$$

Since we already know that $(x-1)$ is a factor of this polynomial, we can find the other factor by inspection. This gives,

$$(x-1)\left((m^2+1)x - (1-m^2)\right) = 0$$

so that

$$x = 1 \text{ or } x = \frac{1-m^2}{1+m^2}.$$

Substituting $x = \frac{1-m^2}{1+m^2}$ into the equation of the line gives

$$\begin{aligned} y &= \frac{1}{m} - \frac{x}{m} \\ &= \frac{1}{m} - \frac{1-m^2}{m(1+m^2)} \\ &= \frac{1+m^2}{m(1+m^2)} - \frac{1-m^2}{m(1+m^2)} \\ &= \frac{2m^2}{m(1+m^2)} \\ &= \frac{2m}{1+m^2} \end{aligned}$$

Therefore the other point of intersection is

$$\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right).$$

b Line L_2 is perpendicular to the line $y = mx$ and so has gradient $-\frac{1}{m}$. Moreover, it goes through the point $(0, 1)$. Therefore, its equation can be easily found:

$$y - 1 = -\frac{1}{m}(x - 0)$$
$$y = 1 - \frac{x}{m}$$

To find where the line intersects the unit circle, we substitute $y = 1 - \frac{x}{m}$ into the equation for the circle, $x^2 + y^2 = 1$ and solve. This gives,

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(1 - \frac{x}{m}\right)^2 &= 1 \\ x^2 + 1 - \frac{2x}{m} + \frac{x^2}{m^2} &= 1 \\ m^2x^2 + m^2 - 2mx + x^2 &= m^2 \\ (1 + m^2)x^2 - 2mx &= 0. \end{aligned}$$

We factorise this expression to give

$$x\left((1 + m^2)x - 2m\right) = 0$$

so that

$$x = 0 \text{ or } x = \frac{2m}{1 + m^2}.$$

Substituting $x = \frac{2m}{1 + m^2}$ into the equation of the line gives

$$\begin{aligned} y &= 1 - \frac{x}{m} \\ &= 1 - \frac{2m}{m(1 + m^2)} \\ &= 1 - \frac{2}{(1 + m^2)} \\ &= \frac{1 + m^2}{1 + m^2} - \frac{2}{(1 + m^2)} \\ &= \frac{m^2 - 1}{1 + m^2} \end{aligned}$$

Therefore, the other point of intersection is

$$\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right).$$

c When reflected in the line $y = mx$, the point $(1, 0)$ maps to

$$\left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$$

while the point $(0, 1)$ maps to

$$\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right).$$

We write these points as the columns of a matrix to give,

$$\begin{bmatrix} \frac{1 - m^2}{1 + m^2} & \frac{2m}{1 + m^2} \\ \frac{2m}{1 + m^2} & \frac{m^2 - 1}{1 + m^2} \end{bmatrix}.$$